The magnetohydrodynamic flow past a flat plate

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(Received 6 October 1958)

The uniform steady flow of an incompressible, viscous, electrically conducting fluid is distorted by the presence of a symmetrically oriented semi-infinite flat plate. The ambient magnetic field is coincident with the ambient velocity field. The description of the resulting fields depends on the physical co-ordinates measured in units of Reynolds number and on the two parameters $\epsilon = \sigma\mu\nu$ and $\beta = \mu H^2/\rho v^2$. This description of the fields is approximated in three different ways and essentially covers the full range of ϵ and β. In particular, when $\beta \geqslant 1$, no steady flow which is uniform at large distances from the plate exists.

1. Introduction

In this paper, we consider the flow of a viscous incompressible electrically conducting fluid of constant properties past a semi-infinite rigid plate. The applied magnetic field is uniform and is directed in the free-stream direction which is parallel to the plane of the plate. This problem is of interest primarily because the magnetic and velocity fields can be described explicitly and accurately as functions of the parameters and some insight into the nature of magnetohydrodynamic flows can be achieved. The flow past a small finite flat plate under the same conditions is also discussed.

Several techniques are used to treat the problems. One successful technique is a direct extension of the asymptotic method which leads to the classical Blasius result; another is the modified Oseen technique (Lewis & Carrier 1949). Of particular interest is the fact that a formal perturbation series in ϵ, the ratio of kinematic to magnetic diffusivities, cannot succeed in the two-dimensional problem. Such a formal expansion fails to exist, just as the expansion of the stream function as a perturbation series in the Reynolds number for the corresponding two-dimensional fluid problem fails to exist.

2. The basic equations

The laws governing the conservation of mass and momentum and those governing the electrodynamics of a steadily moving incompressible viscous electrically conducting fluid of constant properties are

$$\nabla^* . \mathbf{q}^* = 0, \tag{2.1}$$

$$(\mathbf{q}^* . \nabla^*)\, \mathbf{q}^* = -\nabla^*\left(\frac{P^*}{\rho^*}\right) + \nu^*\Delta^*\mathbf{q}^* + \frac{\mu^*}{\rho^*}(\mathbf{j}^* \times \mathbf{H}^*), \tag{2.2}$$

$$j^* = \sigma^*[\mathbf{E}^* + \mu(\mathbf{q}^* \times \mathbf{H}^*)], \tag{2.3}$$

$$\nabla^* \times \mathbf{H}^* = \mathbf{j}^*, \tag{2.4}$$

$$\nabla^* . \mathbf{H}^* = \nabla^* \times \mathbf{E}^* = \nabla^* . \mathbf{E}^* = 0, \tag{2.5}$$

where the symbols are defined as follows: $\mathbf{H}^*$, magnetic field intensity; $\mathbf{E}^*$, electric field; $\mathbf{j}^*$, current density; σ^*, electrical conductivity; μ^*, magnetic permeability; $\mathbf{q}^*$, fluid velocity; P^*, pressure; ρ^*, density; ν^*, kinematic viscosity. The two-dimensional geometry we shall consider here is such that the magnetic field $\mathbf{H}^*$ and the fluid velocity $\mathbf{q}^*$ are each perpendicular to the z-direction. The plate lies in the $y = 0$ plane with its front edge at $x = 0$. The induced current $\mathbf{j}^*$ is then in the z-direction. The electric field can be taken to be zero, $\mathbf{E}^* = 0$, since this choice is rigorously consistent with the foregoing equations and geometry. It is an interesting choice in that it corresponds to the axially directed flow past a semi-infinite pipe of radius R in that limiting case for which $R \to \infty$. In other words, the 'points' (x, y, ∞) and $(x, y, -\infty)$ are short circuited. We introduce the dimensionless variables

$$x^* = \frac{\nu^*}{v_0^*} x, \quad y = \frac{\nu^*}{v_0^*} y, \quad \mathbf{H}^* = H_0^* \mathbf{H}, \quad \mathbf{q}^* = v_0^* \mathbf{q}, \quad P^* = \rho^* v_0^{*2} P, \quad \mathbf{j}^* = H_0^* \frac{v_0^*}{\nu^*} j.$$

If, in addition, we use equation (2.4) to eliminate $\mathbf{j}$ from (2.2), there results

$$\nabla . \mathbf{q} = 0, \tag{2.6}$$

$$(\mathbf{q} . \nabla)\, \mathbf{q} = -\nabla(P + \beta/2\mathbf{H} . \mathbf{H}) + \nabla^2 \mathbf{q} + \beta(\mathbf{H} . \nabla)\, \mathbf{H}, \tag{2.7}$$

$$\nabla \times \mathbf{H} = \epsilon \mathbf{q} \times \mathbf{H}, \tag{2.8}$$

$$\nabla . \mathbf{H} = 0, \tag{2.9}$$

where $\epsilon = \sigma^* \nu^* \mu^*$ and $\beta = \mu^*(H_0^*)^2/\rho^*(v_0^*)^2$. In terms of a magnetic potential A_0, such that $\mathbf{H} = \nabla \times [A_0(x, y)\, \mathbf{i}_3]$ and a stream function ψ_0 such that $\mathbf{q} = \nabla \times [\psi_0(x, y)\, \mathbf{i}_3]$, equations (2.6) to (2.9) can be reduced to

$$\Delta\Delta\psi_0 - \psi_{0_y} \Delta\psi_{0_x} + \psi_{0_x} \Delta\psi_{0_y} + \beta[A_{0_y} \Delta A_{0_x} - A_{0_x} \Delta A_{0_y}] = 0, \tag{2.10}$$

$$\Delta A_0 - \epsilon(\psi_{0_y} A_{0_x} - \psi_{0_x} A_{0_y}) = 0, \tag{2.11}$$

where Δ denotes the Laplacian operator and the suffices x and y denote differentiation.

The proper asymptotic treatment of the classical problem in which $\beta = 0$ and in which equation (2.8) does not appear reveals that the stream function for that problem, ψ_1, can be written in terms of the parabolic co-ordinates $\zeta = \xi + i\eta = (x + iy)^{\frac{1}{2}}$ as

$$\psi_1 \sim \xi F_0(\eta) + \mathscr{R}\left(\frac{\ln \zeta}{\zeta} F_{11}(\eta)\right) + \mathscr{R}\left(\frac{1}{\zeta} F_1(\eta)\right) + \dots.$$

Assuming that the asymptotic solution of the present problem is of the same form (as it must be), we write

$$\psi_0 \sim \xi f(\eta) + \dots, \tag{2.12}$$

$$A_0 \sim \xi g(\eta) + \dots. \tag{2.13}$$

The substitution of these equations into equations (2.10) and (2.11) (properly written in parabolic co-ordinates) leads to

$$f''' + ff'' - \beta gg'' = O(\xi^{-1}), \tag{2.14}$$

$$g'' + \epsilon(fg' - gf') = O(\xi^{-1}). \tag{2.15}$$

The foregoing equations, in whichever form we adopt them, are highly non-linear, and explicit solutions cannot be anticipated. However, their form strikingly resembles the equations which arise when one describes the diffusion and convection of vorticity and heat in a moving viscous fluid. In that problem, ψ_0 would be the vorticity and A_0 the temperature. The only term which differs in the two problems is the final parenthesis of equation (2.10) or the last term on the left-hand side of equation (2.14). For such heat and vorticity transport problems a modification of the Oseen linearization has been shown to be surprisingly successful in a very large variety of problems. The details of this linearization will depend on the applied magnetic field and can best be introduced when the particular boundary value problems are discussed.

3. The uniform magnetic field problem

We shall consider first the flow field which ensues when the applied magnetic field is a uniform field, H_0^*, in the x-direction; the plate occupies the half plane $y = 0$, $x > 0$ and we seek solutions of equations (2.6) through (2.9) with $\mathbf{q} = \mathbf{i}_2 . \mathbf{H} = 0$ on the plate, $\mathbf{H} \to \mathbf{i}_1$, $\mathbf{q} \to \mathbf{i}_1$ as $x \to -\infty$. In terms of ψ_0 and A_0 the boundary conditions are $\psi_{0_y} \to 1$, $\psi_{0_x} \to 0$, $A_{0_y} \to 1$, $A_{0_x} \to 0$ as $x \to -\infty$ with $\psi_0(x, 0) = A_{0_y}(x, 0) = 0$ and $\psi_{0_y}(x, 0) = 0$ for $x \geqslant 0$.

Let $\mathbf{q} = \mathbf{i}_1 + \mathbf{v}$ and $\mathbf{H} = \mathbf{i}_1 + \mathbf{h}$ and replace the non-linear terms $(\mathbf{q} . \nabla)\, \mathbf{q}$, $(\mathbf{H} . \nabla)\, \mathbf{H}$ and $\mathbf{q} \times \mathbf{H}$ in equations (2.6) and (2.7) by $\partial \mathbf{v}/\partial x$, $\partial \mathbf{h}/\partial x$ and $\mathbf{i}_1 \times \mathbf{h} - \mathbf{i}_1 \times \mathbf{v}$, respectively. This linearization represents an extension of Oseen's treatment of viscous flows to the magnetohydrodynamic flow. Physically, the convective velocity and the 'convective' magnetic field contributions are replaced by their free-stream values (that is, by the velocity and magnetic fields which would be present if the obstacle were absent). In fact, the most effective replacement of these terms requires that we use appropriate averages of the x component of velocity and the x component of the magnetic field instead of the free-stream values. We shall not discuss here what these averages are but they can be found readily by the same method that was used in treating conventional boundary layer problems (Carrier 1959). Note that if these averages were used, the mathematical problem to be solved would be *identical* with that treated here. Only the definition of x, y, β, ϵ would be changed and these by constant factors of order unity. The linearized forms of equations (2.10) and (2.11) are

$$\Delta(\Delta\psi - \psi_x + \beta A_x) = 0, \tag{3.1}$$

$$\Delta A + \epsilon(\psi_x - A_x) = 0, \tag{3.2}$$

where $A = -y + A_0$ and $\psi = -y + \psi_0$ ($\mathbf{v} = \nabla \times \psi \mathbf{i}_3$, $\mathbf{h} = \nabla \times A\mathbf{i}_3$). The boundary conditions are $A_x = \psi_x = 0$ for $y = 0$, $\psi_y = -1$ for $y = 0, x > 0$, and A_x, A_y, ψ_x, ψ_y all approach zero far from the plate. It is convenient to write the boundary condition on ψ_y as the limit as $\delta \to 0$ of $\psi_y(x, o) = -e^{-\delta x}$ for $x > 0$. The functions ψ, ψ_y, ψ_{yyy} are continuous across the plate; ψ_{yy}, however, will be discontinuous. Define $f(x)$ to be the discontinuity of ψ_{yy} across the plate,

$$f(x) = (\psi_{yy})_{y=0+} - (\psi_{yy})_{y=0-} = [\psi_{yy}]_{y=0}, \tag{3.3}$$

and note that $f(x) = 0$ for $x < 0$. The solution of this boundary value problem is obtained by Fourier transform techniques. Define the double Fourier transforms

$$\bar{A}(p, r) = \iint_{-\infty}^{\infty} A(x, y) \exp(-i[px + ry])\, dx\, dy, \tag{3.4}$$

and

$$\bar{\psi}(p, r) = \iint_{-\infty}^{\infty} \psi(x, y) \exp(-i[px + ry])\, dx\, dy. \tag{3.5}$$

In particular the y transform is defined as an integral form $-\infty$ to $0-$ plus an integral from $0+$ to $+\infty$. If $\bar{f}(p) = \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx$, then equations (3.1) and (3.2) become

$$(p^2 + r^2)[(p^2 + r^2 + ip)\bar{\psi} - ip\beta\bar{A}] = ir\bar{f}, \tag{3.6}$$

$$(p^2 + r^2 + ip\epsilon)\bar{A} = i\epsilon p\bar{\psi}, \tag{3.7}$$

so that

$$\bar{\psi}(p, r) = ir\bar{f}(p)\frac{p^2 + r^2 + i\epsilon p}{p^2 + r^2}[(p^2 + r^2 + ip)(p^2 + r^2 + i\epsilon p) + \epsilon\beta p^2]^{-1}, \tag{3.8}$$

and

$$\bar{A} = \frac{i\epsilon p\bar{\psi}}{p^2 + r^2 + i\epsilon p} = \frac{-\epsilon rp\bar{f}}{p^2 + r^2}[(p^2 + r^2 + ip)(p^2 + r^2 + i\epsilon p) + \epsilon\beta p^2]^{-1}. \tag{3.9}$$

Define the function

$$\bar{\phi}(p, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\psi}(p, r)\, e^{iry}\, dr = \int_{-\infty}^{\infty} \psi(x, y)\, e^{-ipx}\, dx. \tag{3.10}$$

Upon substituting for $\bar{\psi}$ from equation (3.10) and performing the contour integration, we find that for $\beta < 1$

$$\frac{y}{|y|}\bar{\phi}(p, y) = \frac{-i\bar{f}(p)}{2p}\left\{\frac{\exp - |y|\,|p|}{\beta - 1} + \frac{1}{\lambda}\left(\frac{1 - \epsilon + \lambda}{1 + \epsilon + \lambda}\right)\exp - |y|\left[p^2 + ip\left(\frac{1 + \epsilon + \lambda}{2}\right)\right]^{\frac{1}{2}} - \frac{1}{\lambda}\left(\frac{1 - \epsilon - \lambda}{1 + \epsilon - \lambda}\right)\exp - |y|\left[p^2 + ip\left(\frac{1 + \epsilon - \lambda}{2}\right)\right]^{\frac{1}{2}}\right\}, \tag{3.11}$$

where $\lambda = [(1 - \epsilon)^2 + 4\epsilon\beta]^{\frac{1}{2}}$.

Anticipating difficulties with the inverse Fourier transform of this function, we consider it to be the limit as $k \to 0$ of

$$\frac{y}{|y|}\bar{\phi}(p, y) = \frac{-i\bar{f}}{2(p - ik)}\left\{\frac{\exp - |y|(p^2 + k^2)^{\frac{1}{2}}}{\beta - 1} + \frac{1}{\lambda}\left(\frac{1 - \epsilon + \lambda}{1 + \epsilon + \lambda}\right) \times \exp - |y|\left[(p - ik)\left(p + i\left(\frac{1 + \epsilon + \lambda}{2}\right)\right)\right]^{\frac{1}{2}} - \frac{1}{\lambda}\left(\frac{1 - \epsilon - \lambda}{1 + \epsilon + \lambda}\right)\exp - |y|\left[(p - ik)\left(p + i\left(\frac{1 + \epsilon - \lambda}{2}\right)\right)\right]^{\frac{1}{2}}\right\}. \tag{3.12}$$

The roots are selected so that the exponentials decay as $|y| \to \infty$. In this way, the Fourier transform $\phi(p, y)$ is an analytic function within some strip in the p-plane containing the real axis, and, therefore, the inverse Fourier transform exists. In particular, we shall perform the limiting process (setting $k = 0$) whenever it is convenient and proper to so do.

In order to determine $\psi(x,y)$ and $f(x)$, $\overline{\phi}_y(p, \mathrm{o})$ must be calculated. It follows from (3.12), after some algebraic manipulation, that

$$\overline{\phi}_y(p,\mathrm{o}) = -\frac{\bar{f}}{4\lambda(p-ik)^{\frac{1}{2}}} \times \left[\frac{\lambda-1+\epsilon}{(p+ik)^{\frac{1}{2}}+\left(p+i\left(\frac{1+\epsilon-\lambda}{2}\right)\right)^{\frac{1}{2}}}+\frac{\lambda+1-\epsilon}{(p+ik)^{\frac{1}{2}}+\left(p+i\left(\frac{1+\epsilon+\lambda}{2}\right)\right)^{\frac{1}{2}}}\right]. \quad (3.13)$$

Let
$$\psi_y(x,\mathrm{o}) = u_0(x)+u_1(x), \quad (3.14)$$
where
$$u_0(x) = -e^{-\delta x} \quad (x>0),$$
$$u_0(x) = 0 \quad (x<0),$$

and $u_1(x) = 0\ (x > 0)$. For $x < 0$, $u_1(x)$ is to be determined as part of the solution of the problem. The Fourier transforms of these functions are

$$\overline{u}_0(p) = -\int_0^\infty e^{-\delta x}\,e^{-ipx}\,dx = i/(p-i\delta) \quad \text{and} \quad \overline{u}_1(p) = \int_{-\infty}^0 u_1(x)\,e^{-ipx}\,dx.$$

Note that $\overline{u}_0(p)$ is an analytic function of the complex variable p in a lower half plane which includes the real axis; similarly, $\overline{u}_1(p)$ is an analytic function in an upper half plane which also includes the real axis. From equation (3.10) it follows that

$$\overline{\phi}_y(p,\mathrm{o}) = \overline{u}_0(p)+\overline{u}_1(p) = \frac{i}{p-i\delta}+\overline{u}_1(p), \quad (3.15)$$

where $\overline{\phi}_y(p,\mathrm{o})$ is analytic within some strip containing the real axis. Combining (3.15) with (3.13), we obtain the relationship

$$\frac{i}{p-i\delta}+\overline{u}_1(p) = -\frac{\bar{f}(p)}{4\lambda(p-ik)^{\frac{1}{2}}}G(p), \quad (3.16)$$

where

$$G(p) = \left[\frac{\lambda-1+\epsilon}{(p+ik)^{\frac{1}{2}}+\left(p+i\left(\frac{1+\epsilon-\lambda}{2}\right)\right)^{\frac{1}{2}}}+\frac{\lambda+1-\epsilon}{(p+ik)^{\frac{1}{2}}+\left(p+i\left(\frac{1+\epsilon+\lambda}{2}\right)\right)^{\frac{1}{2}}}\right], \quad (3.17)$$

and from this we can find $\overline{u}(p)$ and $\bar{f}(p)$ by the Wiener–Hopf method. Since $f(x) = 0$ for $x < 0$, $\bar{f}(p)$ is analytic in a lower half plane containing the real axis; it is evident that $G(p)$ is analytic in an upper half plane. Moreover, $G(p)$ has no zero's in the Riemann sheet under consideration. Note that the kernel function was 'split' in the manipulation that led to (3.13). If we label the functions which appear in (3.16) with a $\oplus$ or $\ominus$ according to whether they are analytic in the upper or lower half plane, we find that

$$\frac{i}{(p-i\delta)_\ominus}+\overline{u}_1(p)_\oplus = -\frac{\bar{f}(p)_\ominus}{4\lambda(p-ik)^{\frac{1}{2}}_\ominus}G(p)_\oplus, \quad (3.18)$$

or

$$\frac{i}{(p-i\delta)_\ominus}\frac{1}{G(p)_\oplus}+\left[\frac{\overline{u}_1(p)}{G(p)}\right]_\oplus = -\left[\frac{\bar{f}(p)}{4\lambda(p-ik)^{\frac{1}{2}}}\right]_\ominus.$$

The first function can be written as the sum of a ⊕ function and a ⊖ function,

$$\frac{i}{(p-i\delta)_{\ominus}}\frac{1}{G(p)_{\oplus}} = \left[\frac{i}{p-i\delta}\left(\frac{1}{G(p)}-\frac{1}{G(i\delta)}\right)\right]_{\oplus} + \frac{i}{G(i\delta)}\frac{1}{(p-i\delta)_{\ominus}}. \tag{3.19}$$

Equation (3.18) can then be rewritten as

$$\left[-\frac{\bar{f}(p)}{4\lambda(p-ik)^{\frac{1}{2}}}-\frac{i}{G(i\delta)}\frac{1}{(p-i\delta)}\right]_{\ominus} = \left[\frac{\bar{u}_1(p)}{G(p)}-\frac{i}{p-i\delta}\left(\frac{1}{G(p)}-\frac{1}{G(i\delta)}\right)\right]_{\oplus}. \tag{3.20}$$

Since the left-hand side is an analytic function in some lower half plane while the right-hand side is analytic in some overlapping upper half plane, the functions

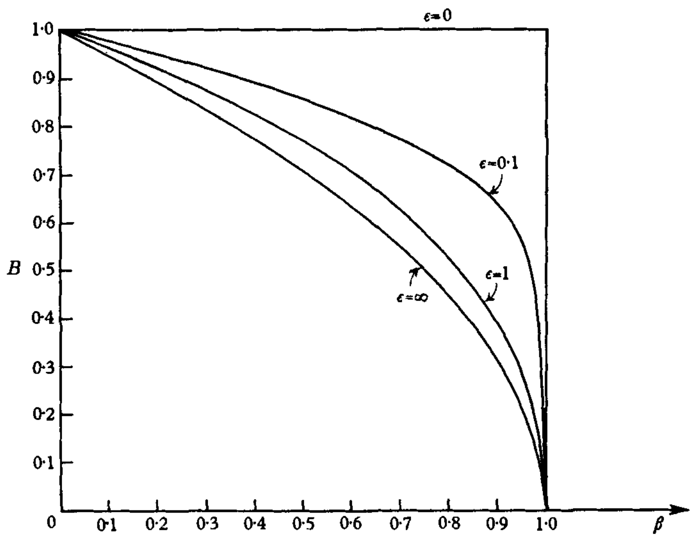

FIGURE 1. The ratio, B, of magnetohydrodynamic skin friction to 'Oseen' skin friction versus β for $\epsilon = 0,\ 1,\ 1,\ \infty$.

are the analytic continuations of each other and are partial representations of one function, $Q(p)$, which is analytic and bounded in the entire plane. This function must be a constant; the constant is readily seen to be zero. Therefore

$$-\frac{\bar{f}(p)}{4\lambda(p-ik)^{\frac{1}{2}}} = \frac{i}{G(0)}\frac{1}{p-i\delta}, \tag{3.21}$$

and upon performing the inverse transform we find that

$$f(x) = 2B(\pi x)^{-\frac{1}{2}}, \tag{3.22}$$

where
$$B = 2\lambda\left[(\lambda-1+\epsilon)\left(\frac{1+\epsilon-\lambda}{2}\right)^{-\frac{1}{2}}+(\lambda+1-\epsilon)\left(\frac{1+\epsilon+\lambda}{2}\right)^{-\frac{1}{2}}\right]^{-1} \tag{3.23}$$

and $\lambda = \{(1-\epsilon)^2+4\epsilon\beta\}^{\frac{1}{2}}$. Except for the factor B, plotted in figure 1, this is the classical result. Note that as $\beta \to 1$, $B \to 0$ and $f(x) \to 0$, which in fact states that the velocity gradient at the plate (hence the skin friction) approaches zero as the applied magnetic field intensity is increased. For $\epsilon = \infty$, $B = (1-\beta)^{\frac{1}{2}}$. The critical point $\beta = 1$ occurs when the magnetic energy contained per unit volume is equal to kinetic energy density, i.e. when $\mu^* H_0^{*2} = \rho^* v_0^{*2}$. We shall see that not only is $f(x) = 0$ at $\beta = 1$, but the total magnetic field **H** as well as the fluid velocity **q** are both identically zero throughout the entire flow. The induced field stops or 'plugs' the flow.

Having determined $f(p)$, it follows that

$$\bar{\phi}(p,y) = -\frac{B(i(p-ik))^{\frac{1}{2}}}{p(p-\delta i)\,\epsilon(1-\beta)}\left[\frac{1-(\epsilon+\lambda)^2}{4\lambda}\left\{\exp - y(p^2+k^2)^{\frac{1}{2}} - \exp - y\left[(p-ik)\left(p+i\left(\frac{1+\epsilon-\lambda}{2}\right)\right)\right]^{\frac{1}{2}}\right\} - \left(\frac{1-(\epsilon-\lambda)^2}{4\lambda}\right)\left\{\exp - y(p^2+k^2)^{\frac{1}{2}} - \exp - y\left[(p-ik)\left(p+i\left(\frac{1+\epsilon-\lambda}{2}\right)\right)\right]^{\frac{1}{2}}\right\}\right], \tag{3.24}$$

where, because of the symmetry, we need only discuss the solutions for $y > 0$. The stream function $\psi(x,y)$ can be determined by an inverse Fourier transform and the fact that

$$\frac{(i)^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty}\frac{(p-ik)^{\frac{1}{2}}}{p(p-\delta i)}e^{ipx}\{\exp - y(p^2+k^2)^{\frac{1}{2}} - \exp - y[(p-ik)(p+is)]^{\frac{1}{2}}\}\,dp = \sqrt{s}\left\{-y+2\xi\left(\eta\,\mathrm{erf}\,\eta\sqrt{s} - \frac{1}{\sqrt{(\pi s)}}[1-\exp(-s\eta^2)]\right)\right\}, \tag{3.25}$$

where k and δ are both set equal to zero, and ξ and η are parabolic co-ordinates, defined by

$$(\xi+i\eta)^2 = x+iy. \tag{3.26}$$

The total stream function $\psi_0 = \psi + y$ is then found to be

$$\psi_0 = \xi f(\eta) = \frac{\lambda+\epsilon-1}{(\lambda+\epsilon-1)\,\omega+(\lambda+1-\epsilon)\,\Omega}\,2\xi\omega\left\{\eta\,\mathrm{erf}\,\eta\Omega - \frac{1-\exp(-\Omega^2\eta^2)}{\Omega\pi^{\frac{1}{2}}}\right\} + 2\xi\frac{\lambda+1-\epsilon}{(\lambda+\epsilon-1)\,\omega+(\lambda+1-\epsilon)\,\Omega}\,\Omega\left\{\eta\,\mathrm{erf}\,\eta\omega - \frac{1-\exp(-\omega^2\eta^2)}{\omega\pi^{\frac{1}{2}}}\right\}, \tag{3.27}$$

where
$$\Omega = \left(\frac{1+\epsilon-\lambda}{2}\right)^{\frac{1}{2}}, \quad \omega = \left(\frac{1+\epsilon+\lambda}{2}\right)^{\frac{1}{2}}. \tag{3.28}$$

For any value of ϵ, and for $\beta < 1$, the flow is uniform at large distances from the plate. As $\epsilon \to 0$ and for $\beta < 1$ equation (3.27) reduces to the classical result obtained by Lewis & Carrier (1949). However, for any non-zero ϵ, $\psi_0 \to 0$ at any given field position ξ, η as $\beta \to 1$; that is, as the applied magnetic field strength is increased to its critical value. At $\beta = 1$, the entire flow is brought to rest.

The magnetic potential A can also be computed by inversion of the transform function $\bar{A}$ so that

$$A = -\frac{2\epsilon}{(\lambda+\epsilon-1)\,\omega+(\lambda+1-\epsilon)\,\Omega}\left\{\Omega\left[-y+2\xi(\eta\,\mathrm{erf}\,\omega\eta) - \frac{(1-\exp-\omega^2\eta^2)}{\omega\pi^{\frac{1}{2}}}\right] + \omega\left[-y+2\xi(\eta\,\mathrm{erf}\,\Omega\eta) - \frac{(1-\exp-\Omega^2\eta^2)}{\Omega\pi^{\frac{1}{2}}}\right]\right\}. \tag{3.29}$$

The total magnetic potential is $A_0 = y + A$. Again as β increases to one, $A_0 \to 0$ for any non-zero ϵ; i.e. the applied magnetic field is completely annulled. For any non-zero ϵ and for $\beta < 1$, $A_0 \to y$ at large distances from the plate. In the limit as $\epsilon \to \infty$ (i.e. the conductivity of the fluid becomes infinite)

$$\psi_0 = A_0 = 2\xi[\eta \operatorname{erf} \eta(1-\beta)^{\frac{1}{2}} - \{1 - \exp - (1-\beta)\eta^2\}(\pi(1-\beta))^{-\frac{1}{2}}],$$

so that the magnetic field is directed along the stream lines and is said to be locked in.

The current density is found to be

$$j_3 = \frac{2\epsilon\omega\Omega(\pi)^{-\frac{1}{2}}}{(\lambda+\epsilon-1)\,\omega+(\lambda+1-\epsilon)\,\Omega}\frac{\xi}{\xi^2+\eta^2}[\exp(-\omega^2\eta^2)-\exp(-\Omega^2\eta^2)]; \quad (3.30)$$

the total current in either the upper half plane or the lower half plane is infinite but these currents are oppositely directed. Again as $\beta \to 1$, the current density becomes zero; in the limit of infinite conductivity

$$j_3 = -\left(\frac{1-\beta}{\pi}\right)^{\frac{1}{2}}\frac{\xi}{\xi^2+\eta^2}\exp - (1-\beta)\,\eta^2.$$

If a uniform magnetic field is applied to the flow past a semi-infinite flat plate, the boundary layer continues to thicken with increase in β until at the critical value $\beta = 1$, the entire flow is plugged. The induced current produces a counter magnetic field which ultimately annuls the entire applied fluid flow and magnetic field. In view of these results it is apparent that no matter how small the conductivity of the fluid, ϵ, any perturbation expansion in ϵ must fail. As long as ϵ is non-zero, the field strength $H_0^* = \left(\frac{\rho^*}{\nu^*}\right)^{\frac{1}{2}} v_0^*$ ($\beta = 1$) will cause plugging.

At first glance, one might have expected that the viscous layer would become thinner when the magnetic field intensity was increased. The superficial argument which leads to that conclusion notes that the stream lines are tilted upward because of the presence of the plate and that they 'pull' the magnetic field lines along with them so that a similar but lesser distortion of the field-line pattern occurs. Since the stream lines are tilted more than the field lines, the force density $(\mathbf{v} \times \mathbf{H}) \times \mathbf{H}$ is directed toward the plate, tending to force any given fluid particle to stay closer to the plate. This argument is incomplete because it fails to account for the induced pressure field which, as it happens, opposes and dominates the above effect. A correct argument which successfully rationalizes the mathematical result requires a discussion of the balance among the diffusion, convection and production (by the electromagnetically applied torque density) of vorticity. One notes, here, that the conservation of angular momentum is governed by

$$\Delta(\nabla \times \mathbf{v}) - (\mathbf{v}\,.\,\mathrm{grad})\,\nabla \times \mathbf{v} + \beta(\mathbf{H}\,.\,\mathrm{grad})\,\nabla \times \mathbf{H} = 0.$$

This equation is just the curl of the momentum equation and $\nabla \times \mathbf{v} = \boldsymbol{\omega}$ is the dimensionless vorticity. Since the distortion of the $\mathbf{H}$ field is similar to that of the $\mathbf{v}$ field, the quantity $(\mathbf{H}\,.\,\mathrm{grad})\nabla \times \mathbf{H}$ is qualitatively like, and of the same sign as, the quantity $(\mathbf{v}\,.\,\mathrm{grad})\,\nabla \times \mathbf{v}$. Consequently, the vorticity production term negates part of the convective contribution and the resulting phenomenon is equivalent to a diffusive-convective problem whose effective convective speed decreases as the magnetic field increases. In such problems this decrease in convective speed implies a thicker diffusion layer and our rationalization is complete.

4. The asymptotic analysis

The most surprising result of the linearized analysis is the fact that both $\mathbf{q}(x, y)$ and $\mathbf{H}(x, y)$ tend to zero for each x, y as $\beta \to 1$ for any ϵ. Clearly, this result warrants further investigation, and we turn our attention to an examination of the asymptotic or Blasius theory, as formulated in § 2, i.e.

$$f''' + ff'' - \beta gg'' = 0, \quad (2.14)$$

$$g'' + \epsilon(fg' - gf') = 0, \quad (2.15)$$

with the boundary conditions $f(0) = f'(0) = g(0) = 0$ and $f'(\infty) = g'(\infty) = 2$. A particularly simple case is that of infinite conductivity, $\epsilon = \infty$, for which the foregoing equations reduce to

$$f''' + (1-\beta) ff'' = 0, \quad (4.1)$$

$$g(\eta) = f(\eta); \quad (4.2)$$

the boundary conditions remain the same as those associated with equations (2.14) and (2.15). The solution of this differential equation is related to the Blasius solution $F(\eta)$ (the solution of (4.1) and boundary conditions for $\beta = 0$) by

$$f(\eta) = (1-\beta)^{-\frac{1}{2}} F[(1-\beta)^{\frac{1}{2}} \eta]. \quad (4.3)$$

In particular, $f''(0) = (1-\beta)^{\frac{1}{2}} F''(0) = 1{\cdot}328(1-\beta)^{\frac{1}{2}}$. The skin friction, in this case, does approach zero as $\beta \to 1$, in exactly the same manner as the linearized theory predicts. The discrepancy between the two is just the magnitude by which the skin friction computed by Oseen differs from that of Blasius. The modified Oseen procedure resolves this difference and brings the two formulae into almost exact agreement.

Equations (2.14) and (2.15) were also integrated numerically and figure 2 presents $f''(0)$ versus β for values of $\epsilon = 0{\cdot}005$, $0{\cdot}05$, 1, 10; the case $\epsilon = \infty$ is the exact result based on the preceding analysis. The surprising result is the occurrence of the $\epsilon = 1$ and $\epsilon = 10$ curves to the left of the 'limit' curve $\epsilon = \infty$, in apparent contradiction with the results based on the linearized analysis (see figure 1). It is expected that the modified Oseen analysis would again resolve these differences but the results of such an analysis are, as yet, lacking. Figures 3 and 4 are plots of the stream lines $\psi_0 = \frac{1}{2}, 1$ and the magnetic potential lines $A_0 = \frac{1}{2}, 1$ as determined by the linear and non-linear theories, respectively, for $\epsilon = 1$, $\beta = \frac{1}{2}$.

To provide additional and more conclusive proof that $\beta = 1$ is indeed critical even when ϵ is finite, we have recourse to the following analysis which clearly indicates the nature of the fields when $1-\beta$ is small. We shall find that an explicit description of both ψ and A can be obtained when $\epsilon = 1$, but that numerical integrations would be needed to detail the results for $\epsilon \neq 1$.

If we introduce the new variables, $z = (1-\beta)^{\frac{1}{2}} \eta$, $\phi(z) = (f - \beta^{\frac{1}{2}} g)(1-\beta)^{-\frac{1}{2}}$, $\Omega(z) = (f + \beta^{\frac{1}{2}} g)(1-\beta)^{\frac{1}{2}}$, equations (2.14) and (2.15) become

$$\Omega''' + \tfrac{1}{2}(1+\epsilon)\,\Omega'' + \tfrac{1}{2}(1-\epsilon)\,\Omega\phi'' = 0, \quad (4.4)$$

$$(1-\beta)\,\phi''' + \tfrac{1}{2}(1+\epsilon)\,\Omega\phi'' + \tfrac{1}{2}(1-\epsilon)\,\phi\Omega'' = 0, \quad (4.5)$$

and, in particular, for $\epsilon = 1$

$$\Omega''' + \phi\Omega'' = 0, \tag{4.6}$$

$$(1-\beta)\phi''' + \Omega\phi'' = 0, \tag{4.7}$$

with $\Omega(o) = 0$, $\Omega \sim 4(z-B)$, $\phi(o) = 0$, $\phi \sim z-B$ and $\Omega'(o) = -(1-\beta)\phi'(o)$, where B is a constant. Let β be close to unity. The form of equations (4.6) and (4.7) suggests that, away from $z = 0$, ϕ and Ω are governed by equation (4.6) and

$$\Omega\phi'' \simeq 0, \tag{4.8}$$

but that, in some neighbourhood of $z = 0$, the seemingly small term of equation (4.7), i.e. $(1-\beta)\phi'''$, is important and a 'boundary layer' phenomenon or 'edge effect' is present (Carrier 1953). We proceed then as follows.

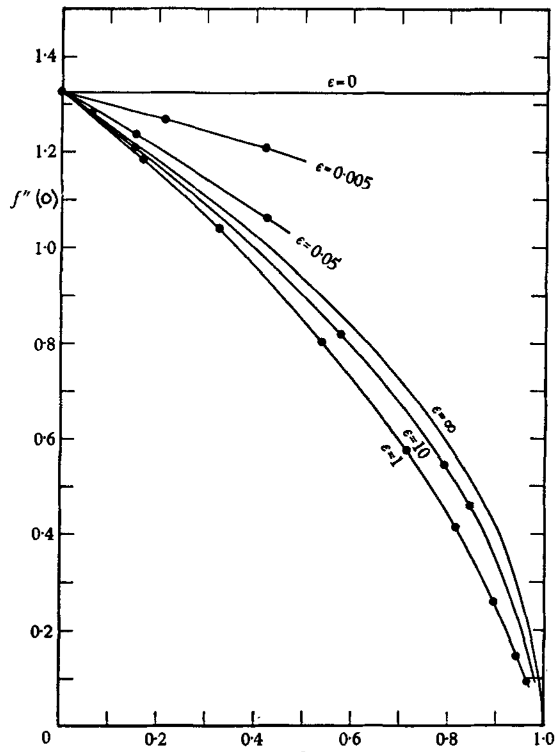

FIGURE 2. 'Skin friction' $f''(o)$ (equation (2.14)) versus β for several values of ϵ.

Let $\Omega'(o) = \lambda$, where λ is some function of β which we can expect to be small, compared to unity, in view of the last boundary condition; we also expect it to be large compared to $1-\beta$. The parameter λ is to be determined in this analysis. Define $\zeta = [\lambda/(1-\beta)]^{\frac{1}{2}} z$ and *assume* that, for $z \ll 1$, $\Omega = \lambda(1-\beta)^{\frac{1}{2}}[F(\zeta)+Q]$, where $Q(\zeta, \beta)$ is small to some as yet undetermined order in $(1-\beta)$ and where F is independent of β. This is equivalent to the assumption that $\Omega''(o)$, $\Omega'''(o)$, ..., are so small that $\Omega'(o)$ is not dominated by $\Omega''(o)z^2$, etc., for any ζ of order unity. If $\Omega''(o)z^2$ *were* the dominant term, a similar procedure could be (and was) followed. However, it turns out that $\Omega''(o)$ is not a dominant contribution and no clarity would be gained by detailing the analysis corresponding to that unsuccessful conjecture.

For $\zeta \gg 1$, equation (4.8) is expected to hold and therefore

$$\phi_i = z - B. \tag{4.9}$$

We can write

$$\phi = \phi_i + B\Phi(\zeta), \tag{4.10}$$

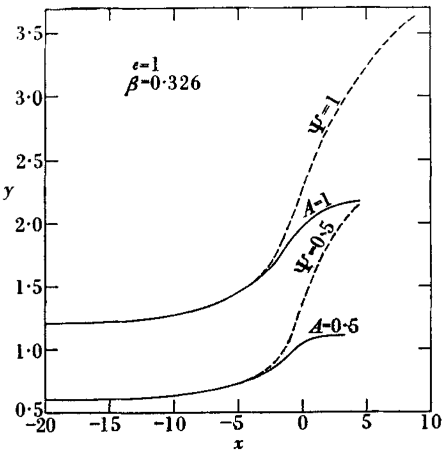

FIGURE 3. The stream lines $\psi_0 = \frac{1}{2}$, 1; and constant potential lines $A_0 = \frac{1}{2}$, 1; for $\epsilon = 1$, $\beta = 0{\cdot}326$ as determined from the linear theory.

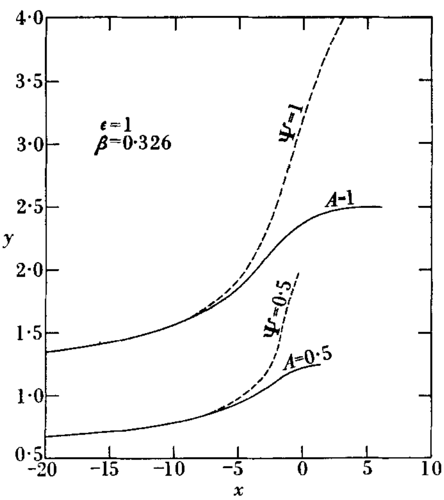

FIGURE 4. The stream lines $\psi_0 = \frac{1}{2}$, 1; and constant potential lines $A_0 = \frac{1}{2}$, 1; for $\epsilon = 1$, $\beta = 0{\cdot}326$ as determined from the non-linear theory.

where $\Phi(o) = 1$ and where Φ must decay to zero very rapidly as $\zeta \to \infty$. Since $\Omega \simeq F(\zeta)$ when $z \ll 1$, equation (4.7) becomes (for $z \ll 1$)

$$\Phi'''(\zeta) + F(\zeta)\,\Phi''(\zeta) = 0, \tag{4.11}$$

and equation (4.6) can be written

$$\Omega'''(z) + [z - B + B\Phi(\zeta)]\,\Omega''(z) = 0, \tag{4.12}$$

so equation (4.12) can be integrated to give

$$\Omega(z) = 4[I(1+\Omega^*)]^{-1}$$
$$\times\left[\int_0^z (z-z')\exp\left\{-\tfrac{1}{2}z'^2+Bz'+[B^2(1-\beta)/\lambda]^{\frac{1}{2}}\int_{\zeta'}^\infty \Phi(\zeta'')\,d\zeta''\right\}dz' + I\Omega^* z\right],$$

where $\zeta' = \zeta(z')$, $\Omega = \lambda/(4-\lambda)$ and

$$I = \int_0^\infty \exp\left\{-\tfrac{1}{2}z'^2+Bz'+[B^2(1-\beta)/\lambda]^{\frac{1}{2}}\int_{\zeta'}^\infty \Phi(\zeta'')\,d\zeta''\right\}dz'$$
$$= \int_0^\infty \exp\{-\tfrac{1}{2}z'^2+Bz'+[B^2(1-\beta)/\lambda]^{\frac{1}{2}}\chi(\zeta')\}\,dz'. \tag{4.13}$$

If the coefficient of χ in (4.13) is of order unity or smaller, then I is closely given by

$$I \simeq \surd(\tfrac{1}{2}\pi)\,(1+\operatorname{erf} B/\surd 2)\,e^{\frac{1}{2}B^2},$$

and if $B \gg 1$ (as it will be when $\beta - 1 \ll 1$),

$$I \simeq \surd(2\pi)\,e^{\frac{1}{2}B^2}.$$

It follows that

$$\Omega''(0) \simeq 4\exp\{[B^2(1-\beta)/\lambda]^{\frac{1}{2}} - \tfrac{1}{2}B^2\} \tag{4.14}$$

and that, for $z \ll 1$,

$$\Omega \simeq [\lambda(1-\beta)]^{\frac{1}{2}}\zeta + 4\exp\{[B^2(1-\beta)/\lambda]^{\frac{1}{2}} - \tfrac{1}{2}B^2\}(1-\beta)\,\zeta^2/\lambda + \ldots,$$

and that, as $z \to \infty$

$$\Omega \sim 4\left(z - B\left\{\left(1+\frac{1}{\surd(2\pi)\,B\,e^{\frac{1}{2}B^2}}\right)(1+\Omega^*)^{-1}\right\}\right)+\ldots.$$

Furthermore

$$\Phi = \frac{\int_\zeta^\infty (\tau-\zeta)\exp\left\{-\int_0^\tau F(\nu)\,d\nu\right\}d\tau}{\int_0^\infty \tau\exp\left\{-\int_0^\tau F(\nu)\,d\nu\right\}d\tau}, \tag{4.15}$$

and

$$\Phi'(0) = \frac{\int_0^\infty \exp\left\{-\int_0^\tau F(\nu)\,d\nu\right\}d\tau}{\int_0^\infty \tau\exp\left\{-\int_0^\tau F(\nu)\,d\nu\right\}d\tau}.$$

In order that $\Omega \sim 4(z-B)$,

$$\Omega^* = \frac{\lambda}{4-\lambda} = [B\,e^{\frac{1}{2}B^2}\surd(2\pi)]^{-1}. \tag{4.16}$$

Furthermore, in order that $\Omega'(0) = -(1-\beta)\,\phi'(0)$,

$$\lambda = -(1-\beta)\,B[\lambda/(1-\beta)]^{\frac{1}{2}}\,\Phi'(0). \tag{4.17}$$

Equations (4.16) and (4.17) imply that

$$(1-\beta)\,B^3 e^{\frac{1}{2}B^2} = 4/\{\surd(2\pi)\,D^2\}^{-1}, \tag{4.18}$$

where $D = -\Phi'(0)$. This defines $B(\beta)$ (except for D, a constant of order unity yet to be determined) and we see that $B \to \infty$ very slowly as $1-\beta \to 0$. The 'stretch factor' in the co-ordinate change is just DB, i.e. $\zeta = DBz$ and $\lambda = B^2D^2(1-\beta)$. Equation (4.14) gives for $\Omega''(0)$,

$$\Omega''(0) = \sqrt{(2\pi)}\, e^{1/D} B^3 D^2 (1-\beta)$$

and
$$F(\zeta) = \zeta + \frac{\sqrt{2\pi}}{D}\int_0^{\zeta}(\zeta-\zeta')\exp\left\{\frac{\zeta}{D}+\frac{1}{D}\int_{\zeta'}^{\infty}\Phi(\zeta'')\,d\zeta''\right\}d\zeta'. \tag{4.19}$$

Equations (4.15) and (4.19) are now equivalent to a non-linear integral equation for $\Phi(\zeta)$ and that equation has a solution for each real positive D. Note that when D is ∞, equation (4.15) implies that $\Phi'(0) = -\sqrt{(\pi/2)}$. On the other hand, as $D \to 0$, $\Phi'(0) \to -\infty$. Since $\Phi'(0)$ is a continuous function of D, the foregoing estimates imply that there is a value of D for which $\Phi'(0) = -D$. This, of course, is the required value for D and the description of the fields is complete when Φ has been calculated for this eigenvalue D. Since D is of order unity and, in particular, is independent of β, one can readily verify that each contribution which was omitted in the foregoing analysis was smaller by a factor $1/B$ than any retained contribution against which it would have been compared had it also been retained. The solution given above, then, is the dominant part of the asymptotic (in $1-\beta$) solution of the problem. Our purpose is now completely served without an integration to find Φ accurately because one readily sees that both f and g as given by the definition preceding (4.4) tend to zero at every meaningful (non-negative) value of η and the flow is 'plugged' as was suggested by the linear theory.

When ϵ is neither zero nor infinite no such elementary treatment can be used. None the less, the problem still displays a boundary layer character similar to that of the $\epsilon = 1$ case, and there is little doubt that this problem (equations (4.4) and (4.5)) has a solution which again implies an $f(\eta)$ and $g(\eta)$ which tend to zero at all η as $\beta \to 1$.

5. Finite flat plate

The magnetohydrodynamic flow past a finite plate of length l^* exists not only for $\beta < 1$ but also for $\beta > 1$. We consider first $\beta < 1$, since for this case the analysis of §3 through equation (3.13) is still valid. If $R = l^*\nu^*/v_0^*$ is the dimensionless plate length, then $f(x) = 0$ for $x > R$ (see (3.3)). The Wiener–Hopf method can no longer be used; to determine $f(x)$ an integral equation can be formulated and solved. The Fourier inversion of (3.13) and use of the convolution theorem yield the result

$$\psi_y(x,0) = -\int_0^R f(t)\left[\frac{\lambda+\epsilon-1}{4\lambda}I(x-t,\Omega^2)+\frac{\lambda+1-\epsilon}{4\lambda}I(x-t,\omega^2)\right]dt, \tag{5.1}$$

where
$$\Omega = \left(\frac{1+\epsilon-\lambda}{2}\right)^{\frac{1}{2}}, \quad \omega = \left(\frac{1+\epsilon+\lambda}{2}\right)^{\frac{1}{2}} \quad (\beta < 1),$$

and
$$I(x,a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}\,dx}{(p^2+k^2)^{\frac{1}{2}}+[(p-ik)(p+ia)]^{\frac{1}{2}}}. \tag{5.2}$$

For $0 < x < R$, $\psi_y(x,0) = -1$ and the resultant integral equation for $f(x)$ is

$$1 = \int_0^R f(t)\left[\frac{\lambda+\epsilon-1}{4\lambda}I(x-t,\Omega^2)+\frac{\lambda+1-\epsilon}{4\lambda}I(x-t,\omega^2)\right]dt. \tag{5.3}$$

The integral in equation (5.2) can be evaluated easily by some simple, but carefully executed, contour integrations and it is found that

$$\left.\begin{aligned} I(x,a) &= \frac{1}{\pi x a}\int_0^{\frac{1}{2}(xa)} K_0(|z|)\,e^z\,dz, \\ I(x,a) &= -\frac{1}{\pi x a}+\frac{e^{\frac{1}{2}(xa)}}{2\pi}\left[K_0\left(\tfrac{1}{2}|ax|\right)-\frac{2}{a}\frac{d}{dx}K_0\left(\tfrac{1}{2}|xa|\right)\right], \\ I(x,a) &= -\frac{1}{\pi x a}+\frac{e^{\frac{1}{2}(xa)}}{2\pi}\left[K_0\left(\tfrac{1}{2}|xa|\right)+\frac{x}{|x|}K_1\left(\tfrac{1}{2}|xa|\right)\right], \end{aligned}\right\} \tag{5.4}$$

where $K_0(z)$ and $K_1(z)$ are modified Bessel functions of the second kind. Another method of deriving this result is presented in the Appendix.

For the first example, let the conductivity of the fluid be infinite, so that

$$\lim_{\epsilon\to\infty}\frac{\lambda+1-\epsilon}{4\lambda}I(x-t,\omega^2) = 0$$

and

$$\lim_{\epsilon\to\infty}\frac{1+\epsilon-\lambda}{4\lambda}I(x-t,\Omega^2) = \tfrac{1}{2}I(x-t,1-\beta).$$

Equation (3.3) reduces to

$$1 = \int_0^R \tfrac{1}{2}(f(t))\,I(x-t,1-\beta)\,dt. \tag{5.5}$$

Henceforth, let the Reynolds number R be very small, $R \ll 1$. The kernel of the integral equation (5.5) can then be approximated by the first few terms of its series expansion

$$\begin{aligned} \tfrac{1}{2}I(x-t,1-\beta) &= -\frac{1}{4\pi}\ln\left(\frac{1-\beta}{4}|x-t|\right)\left[1+\frac{1-\beta}{4}(x-t)\right] \\ &\quad+\frac{1}{4\pi}\left[1+\psi(1)+\frac{1-\beta}{4}(x-t)\,(3\psi(1)-\psi(2))\right]+\ldots. \end{aligned} \tag{5.6}$$

The simplest approximation, for sufficiently small R, is

$$\tfrac{1}{2}I(x-t,1-\beta) = -\frac{1}{4\pi}\ln\left(\frac{1-\beta}{4}|x-t|\right); \tag{5.7}$$

equation (5.5) is then approximated by

$$-1 = \int_0^R \frac{f(t)}{4\pi}\ln\left(\frac{1-\beta}{4}|x-t|\right)dt. \tag{5.8}$$

The solution of this integral equation was obtained by Carlemann (1922) and is

$$f(x) = \frac{-4}{(x(R-x))^{\frac{1}{2}}\ln\{\frac{1}{16}(1-\beta)R\}}. \tag{5.9}$$

Note that as $\beta \to 1, f(x) \to 0$ (the skin friction approaches zero). This immediately implies that ψ and A both become zero at this critical value. We again find that the entire flow can be brought to rest by increasing the strength of the applied magnetic field.

The next approximation would be to use the first two terms of (5.6)

$$\tfrac{1}{2}I(x-t, 1-\beta) \simeq -\frac{1}{4\pi}\ln\left(\frac{1-\beta}{4}|x-t|\right)+\frac{1-\gamma}{4\pi}. \tag{5.10}$$

For this kernel, it is found that

$$f(x) = \frac{-4}{\ln\{\frac{1}{16}(1-\beta)R\}-(1+\gamma)}[x(R-x)]^{-\frac{1}{2}}, \tag{5.11}$$

where γ is the Euler constant. For still higher approximations any finite number of terms of the series expansion in (5.6) can be used. A method for obtaining the

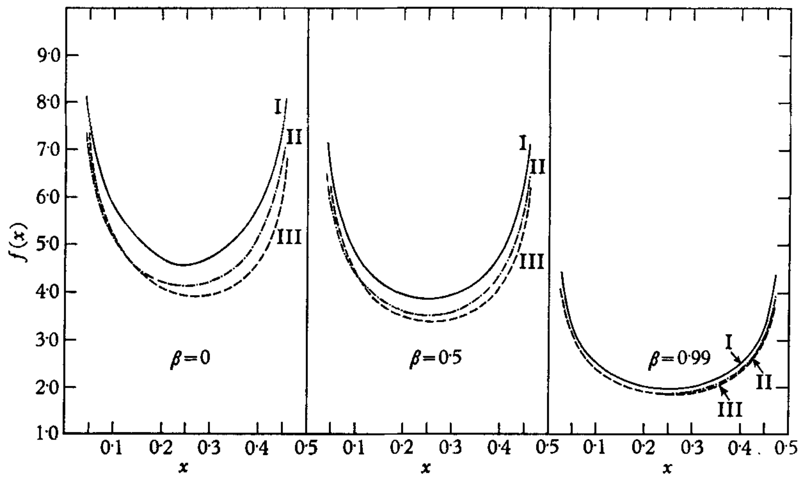

FIGURE 5. 'Skin friction' $f(x)$ versus distance for plate length $R = 0{\cdot}5$ and $\beta = 0, 0{\cdot}5, 0{\cdot}99$; I, solution obtained from equation (5.9); II, equation (5.11); III, equation (5.15).

exact solutions to the corresponding integral equations has been given by Pearson (1957). If three terms of equation (5.6) are used, a sufficiently good approximation (to $O(R^2)$) of the exact solution is

$$f(x) = \frac{-4(x(R-x))^{-\frac{1}{2}}[1+\frac{1}{4}(x-\frac{1}{2}R)(1-\beta)\{\ln\{\frac{1}{16}(1-\beta)R\}+1-\frac{1}{2}(3\psi(1)-\psi(2))\}]}{\ln\{\frac{1}{16}(1-\beta)R\}-(1-\gamma)} \tag{5.12}$$

The solutions, equations (5.9), (5.10) and (5.12), are plotted in Fig. 5 for $\beta = 0$, $0{\cdot}5$, $0{\cdot}99$. The relatively large plate length, $R = 0{\cdot}5$, was chosen to compare the accuracy of the successive approximations. The drag on the plate is given by

$$-2\pi v_c \rho\nu/\ln\{\tfrac{1}{16}(1-\beta)R\}$$

if we use the solution given by (5.9) or

$$-2\pi v_0 \rho\nu/[\ln\{\tfrac{1}{16}(1-\beta)R\}-1+\gamma]$$

if we use either (5.11) or (5.12). The latter formula is in agreement with the results given in Tomotika & Aoi (1953).

The horizontal velocity component on the x-axis can be computed without difficulty by means of equation (5.1). If we use the simplest approximation to $f(x)$ given in (5.9),

$$\psi_y(x, o) = \frac{2}{\pi \ln\{\frac{1}{16}(1-\beta)R\}} \int_0^R [t(R-t)]^{-\frac{1}{2}} \times \left[\frac{-1}{(1-\beta)(x-t)} + \tfrac{1}{2}e^{\frac{1}{2}(1-\beta)(x-t)}\left\{K_0'\left(\frac{1-\beta}{2}|x-t|\right) + \frac{x-t}{|x-t|}K_1\left(\frac{1-\beta}{2}|x-t|\right)\right\}\right] dt. \quad (5.13)$$

To determine $\psi_y(x, o)$ near the plate, we can again approximate the kernel by equation (5.7). The resultant integrals are all tabulated and we find that upstream, near the plate, $|x-t| \ll 1, x > 0$,

$$\psi_y(x, o) \simeq -\frac{\ln\{\frac{1}{8}(1-\beta)\}[(\frac{1}{2}R-x)+(x(x-R))^{\frac{1}{2}}]}{\ln\{\frac{1}{16}(1-\beta)R\}} \quad (5.14)$$

and that downstream, near the plate,

$$\psi_y(x, o) \simeq -\frac{\ln\{\frac{1}{8}(1-\beta)\}[(x-\frac{1}{2}R)+(x(x-R))^{\frac{1}{2}}]}{\ln\{\frac{1}{16}(1-\beta)R\}}. \quad (5.15)$$

At large distances from the plate, we can replace the kernel by its asymptotic values to obtain asymptotic formulae for the function $\psi_y(x, o)$. Upstream, far from the plate,

$$\psi_y(x, o) \sim \frac{2}{(1-\beta)|x|\ln\{\frac{1}{16}(1-\beta)R\}} \quad (5.16)$$

while far downstream

$$\psi_y(x, o) \sim \left[\frac{\pi}{(1-\beta)x}\right]^{\frac{1}{2}} \frac{2}{\ln\{\frac{1}{16}(1-\beta)R\}}. \quad (5.17)$$

It is evident that for $\beta < 1$, the effects of the plate on the fluid are most prominent in the downstream wake.

For a fluid of arbitrary finite conductivity, (5.3) must be solved. This can be done by exactly the same methods just used. For the moment, let $\omega x < 1$ and $\Omega x < 1$. (This implies that $\epsilon = O(1)$.) If we approximate the kernel of (5.3) by the first term of its series expansion the resultant integral equation is

$$-1 = \int_0^R \frac{f(t)}{2\pi}\left[\left(\frac{\lambda+\epsilon-1}{4\lambda}\right)\ln\{\tfrac{1}{4}\Omega^2|x-t|\} + \left(\frac{\lambda+1-\epsilon}{4\lambda}\right)\ln\{\tfrac{1}{4}\omega^2|x-t|\}\right] dt. \quad (5.18)$$

The solution of this equation is

$$f(x) = -4[x(R-x)]^{-\frac{1}{2}}\Lambda^{-1},$$

where

$$\Lambda = \ln\tfrac{1}{16}R + \left(\frac{\lambda+\epsilon-1}{\lambda}\right)\ln\Omega + \left(\frac{\lambda+1-\epsilon}{\lambda}\right)\ln\omega. \quad (5.19)$$

This formula is also valid for very large ϵ, for although $I(\omega x)$ cannot be accurately approximated by the first term of its series expansion, the coefficient of this term, $(\lambda+1-\epsilon)/2\lambda$, is very small. In fact,

$$\lim_{\epsilon\to\infty} \frac{\lambda+1-\epsilon}{2\lambda} I(\omega^2 x) = 0.$$

The solution for $\epsilon = \infty$ represents the first term of the asymptotic expansion of $f(x)$ for very large ϵ. Since the limits of ω^2, Ω^2, $(\lambda+\epsilon-1)/2\lambda$ and $(\lambda+1-\epsilon)/2\lambda$, as $\beta \to 1$, are $1+\epsilon$, 0, $\epsilon/(1+\epsilon)$ and $1/(1+\epsilon)$, respectively, $\beta = 1$ is a critical value, the value at which plugging occurs, and

$$\lim_{\beta\to 1} f(x) = 0.$$

Downstream, near the plate, we find that

$$\Lambda\psi_y(x,o) = -\ln\tfrac{1}{8}[(x-\tfrac{1}{2}R)+(x(x-R))^{\frac{1}{2}}] - \frac{\lambda+\epsilon-1}{\lambda}\ln\Omega - \frac{\lambda+1-\epsilon}{\lambda}\ln\omega, \tag{5.20}$$

whereas far from the plate

$$\Lambda\psi_y(x,o) \sim \left(\frac{\pi}{x}\right)^{\frac{1}{2}}\left[\frac{\lambda+\epsilon-1}{\lambda\Omega}+\frac{\lambda+1-\epsilon}{\lambda\omega}\right]. \tag{5.21}$$

Upstream, near the plate

$$\Lambda\psi_y(x,o) = -\ln\tfrac{1}{8}((R-\tfrac{1}{2}x)+x(x-R)^{\frac{1}{2}}) - \frac{\lambda+\epsilon-1}{\lambda}\ln\Omega - \frac{\lambda+1-\epsilon}{\lambda}\ln\omega \tag{5.22}$$

and far from the plate

$$\Lambda\psi_y(x,o) \sim \frac{1}{x}\left[\frac{\lambda+\epsilon-1}{\lambda\Omega^2}+\frac{\lambda+1-\epsilon}{\lambda\omega^2}\right]. \tag{5.23}$$

The results expressed in (5.14) through (5.17) are obtainable from these by setting $\epsilon = \infty$.

The case $\beta > 1$ requires only a slight modification of technique. Equation (3.11) is still valid when the proper interpretation is placed upon the root quantities. The term $(p^2+\frac{1}{2}ip(1+\epsilon+\lambda))^{\frac{1}{2}}$ is still interpreted as

$$\lim_{k\to 0}(p-ik)(p+\tfrac{1}{2}i(1+\epsilon+\lambda))^{\frac{1}{2}};$$

however, since $\frac{1}{2}(1+\epsilon-\lambda)$ is now negative, and $\Omega_*^2 = \frac{1}{2}(\lambda-1-\epsilon)$, we must now interpret $(p^2-ip\Omega_*^2)^{\frac{1}{2}}$ as

$$\lim_{k\to 0}(p+ik)(p-i\Omega_*^2)^{\frac{1}{2}}.$$

When p divides a quantity containing the first root it is replaced by $p-ik$; when p divides a quantity containing the second root it is replaced by $p+ik$. The analysis is then similar to that for $\beta < 1$ and will not be repeated in detail. It follows that

$$\psi_y(x,o) = -\int_0^R f(t)\left[\frac{\lambda-1+\epsilon}{4\lambda}I(t-x,\Omega_*^2)+\frac{\lambda+1-\epsilon}{4\lambda}I(x-t,\omega^2)\right]dt, \tag{5.24}$$

where $I(x,a)$ is given in equation (5.4). In the interests of simplicity we restrict ourselves to a discussion of the case of infinite conductivity. Equation (5.24) reduces to

$$\psi_y(x,o) = -\int_0^R \tfrac{1}{2}f(t)\,I(t-x,\beta-1)\,dt. \tag{5.25}$$

For sufficiently small x, $I(t-x,\beta-1) = I(x-t,\beta-1)$ as shown by (5.7) and (5.10). To this order, $f(x)$ may be obtained from (5.9) or (5.11) by replacing $1-\beta$ by $\beta-1$

or $|1-\beta|$. The horizontal velocity component $\psi_y(x,0)$ can be determined for small x (near the plate), by making the same replacement in (5.14) and (5.15). The asymptotic behaviour of $\psi_y(x,0)$ for $\beta > 1$, $\epsilon = \infty$ is just the reverse of its behaviour for $\beta < 1, \epsilon = \infty$. Downstream, $(\beta-1)x \gg R$,

$$\psi_y(x,0) \sim \frac{2}{(\beta-1)x\ln\{\frac{1}{16}(\beta-1)R\}}, \tag{5.26}$$

whereas, upstream $x < 0$, $(\beta-1)x \gg R$,

$$\psi_y(x,0) \sim 2\left(\frac{\pi}{(\beta-1)|x|}\right)^{\frac{1}{2}} \frac{1}{\ln\{\frac{1}{16}(\beta-1)R\}}. \tag{5.27}$$

For $\beta > 1, \epsilon = \infty$, the effects of the plate on the fluid are most prominent *upstream* and not in the wake as is the case for $\beta < 1$. The corresponding statement for finite conductivity, $\beta > 1$, is that the effects of the plate on the fluid are *as* prominent upstream as they are downstream. The asymptotic behaviour of $\psi_y(x,0)$ in this case is inversely proportional to $\sqrt{x}$ in either direction. The significance of these results lies in the fact that for $\beta > 1$ the plate is moving at a velocity which is subsonic compared to the Alfvén wave speed $c = (\mu^*/\rho^*)^{\frac{1}{2}} H_0^*$. The disturbance can propagate upstream by means of these waves, whereas, for $\beta < 1$, the plate is moving at a supersonic speed compared to this wave velocity, making it impossible for a disturbance to propagate upstream.

6. Conclusion

The magnetohydrodynamic flow past a flat plate exhibits a rather surprising effect. For any non-zero conductivity there is a single critical applied magnetic field, $H_0^* = (\rho^*/\mu^*)^{\frac{1}{2}} v_0^*$ for which the velocity and magnetic field at any field point, x, y, are zero. In particular, no steady 'subsonic' flow past an infinite plate (the flow being uniform at infinity) can exist; by 'subsonic' we mean a flow in which U_∞ is less than the Alfvén speed. This is somewhat analogous to the situation which arises when one studies the inviscid compressible flow past a wedge, but that analogy is loose because our displacement thickness is parabolic, not wedge-like, and steady subsonic hydrodynamic flow past a parabola can exist.

On the other hand, there is a steady flow past a finite plate (or a more general cylinder) both at $\beta < 1$ and at $\beta > 1$. When $\beta > 1$, however, the upstream influence is much stronger than it is when $\beta < 1$; in fact, the structure of the upstream velocity profile for $\beta > 1$ is made like that of the laminar wake when no magnetic field is present.

Appendix

We now derive an integral expression for the stream function $\psi(x,y)$. We will consider $\epsilon = \infty$ only; the extension of the methods to the case of finite conductivity is straightforward. For $\epsilon = \infty$ and $\beta < 1$, equation (3.11) becomes

$$\phi(p,y) = \frac{f(p)}{(1-\beta)\,2pi}[\exp(-|y|\,|p|) - \exp(-|y|(p^2+ip(1-\beta))^{\frac{1}{2}})]$$

so that

$$\phi_y(p,y) = \frac{f(p)}{2i(1-\beta)}\left[\frac{(p^2+k^2)^{\frac{1}{2}}}{p-ik}\exp[-y(p^2+k^2)^{\frac{1}{2}}] - p^{-\frac{1}{2}}(p+i(1-\beta))^{\frac{1}{2}}\exp[-y(p^2+ip(1-\beta))^{\frac{1}{2}}]\right],$$

where we have again introduced the factor k. This last equation can be reduced to a more convenient form,

$$\phi_y(p,y) = \frac{f(p)}{2(1-\beta)}\left[\frac{(k-ip)}{(p^2+k^2)^{\frac{1}{2}}}\exp\{-|y|(p^2+k^2)^{\frac{1}{2}}\} - \frac{((1-\beta)-ip)}{(p^2+ip(1-\beta))^{\frac{1}{2}}}\exp\{-|y|(p^2+ip(1-\beta))^{\frac{1}{2}}\}\right]$$

Since
$$\psi_y(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\phi_y(p,y)\,dp,$$

application of the convolution theorem yields

$$\psi_y(x,y) = \int_0^R f(t)\,[G(x-t,y) - H(x-t,y)]\,dt,$$

where

$$H(x,y) = -\frac{1}{4\pi(1-\beta)}\int_{-\infty}^{\infty}\frac{(1-\beta-ip)}{(p^2+ip(1-\beta))^{\frac{1}{2}}}\exp\{ipx-y(p^2+ip(1-\beta))^{\frac{1}{2}}\}\,dp$$

and
$$G(x,y) = \frac{1}{4\pi(1-\beta)}\int_{-\infty}^{\infty}\frac{k-ip}{(p^2+k^2)^{\frac{1}{2}}}\exp\{ipx-y(p^2+k^2)^{\frac{1}{2}}\}\,dp.$$

From Foster & Campbell (1948, formula no. 868)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{i\xi x-y(k^2-\xi^2)^{\frac{1}{2}}\}}{(k^2-\xi^2)^{\frac{1}{2}}}\,d\xi = \frac{1}{\pi}K_0(k(x^2+y^2)^{\frac{1}{2}}).$$

It follows that

$$G(x,y) = -\frac{1}{2\pi(1-\beta)}\left(k-\frac{\partial}{\partial x}\right)K_0(k(x^2+y^2)^{\frac{1}{2}}),$$

$$H(x,y) = \left(1-\beta-\frac{\partial}{\partial x}\right)e^{\frac{1}{2}(1-\beta)x}K_0(\tfrac{1}{2}(1-\beta)(x^2+y^2)^{\frac{1}{2}})\frac{1}{2\pi(1-\beta)}$$

and, as $k \to 0$,

$$G(x,y)-H(x,y) = \frac{x}{2\pi(1-\beta)(x^2+y^2)} - \frac{1}{4\pi}e^{\frac{1}{2}(1-\beta)x}K_0(\tfrac{1}{2}(1-\beta)(x^2+y^2)^{\frac{1}{2}}) - \frac{x}{2\pi(x^2+y^2)^{\frac{1}{2}}}K_1(\tfrac{1}{2}(1-\beta)(x^2+y^2)^{\frac{1}{2}}).$$

This reduces to (5.4) upon setting $y = 0$. The stream function is obtained by integrating with respect to y,

$$\psi(x,y) = \int_0^y dz\int_0^R f(t)\{G(x-t,z)-H(x-t,z)\}\,dt.$$

The function $f(x)$ is given in equations (5.9), (5.11) or (5.15). For $\beta > 1$ it is necessary to alter the analysis as indicated at the end of § 5. The modified kernel can then be obtained from this expression for $G(x, y) - H(x, y)$ by replacing $1-\beta$ by $\beta - 1$ and x by $-x$. The function $f(x)$ is determined by making the same replacement in (5.9) and (5.11).

Some of the work reported here was sponsored by the Office of Naval Research under contract Nonr 1866(20).

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